P = pressure, dynes/sq.cm.
r = radial distance, cm., or reduced radial distance (r/R)
r/R = reduced radial distance
R_d = radius of smaller or downstream tube, diameters
R_u = radius of larger or upstream tube, diameters
U = z-component velocity, v_z/U_0
U_0 = area average axial velocity in tube (with one or two exceptions, the downstream tube), cm./sec.
v = velocity, cm./sec.
V = r-component velocity, v_r/U_0
z = axial distance, cm., or reduced axial distance (z/D)
z/D = reduced axial distance
Z_e = entrance length, diameters

Greek Letters

β = ratio of large-tube diameter to small-tube diameter
ξ = transformed axial coordinate
ρ = density, g./cu.cm.
ψ = stream function

Subscripts

r = radial direction
z = axial direction

LITERATURE CITED

1. Astarita, G., and G. Greco, *Ind. Eng. Chem. Fundamentals*, 7, 27 (1968).
2. Atkinson, B., M. P. Brocklebank, C. C. H. Card, and J. M. Smith, *AIChE J.*, **15**, 548 (1969).
3. Burke, J. P., and N. S. Berman, Paper 69-WA/FE-13, presented at the ASME Winter Annual Meeting, Los Angeles (November 16-20, 1969).
4. Campbell, W. D., and J. C. Slattery, *Trans. Am. Soc. Mech. Engrs.*, **D85**, 41 (1963).
5. Carter, T. R., Ph.D. thesis, Univ. of Utah, Salt Lake City (1969); also, paper presented at 66th Natl. Am. Inst. Chem. Engrs. Meeting, Portland, Oregon (August, 1969).
6. Christiansen, E. B., and H. E. Lemmon, *AIChE J.*, **11**, 995 (1965).
7. Collins, M., and W. R. Schowalter, *AIChE J.*, **9**, 98 (1963).
8. Hornbeck, R. W., *Appl. Sci. Res.*, **A13**, 224 (1964).
9. Kelsey, S. J., Ph.D. thesis, Univ. of Utah, Salt Lake City (1971).
10. LaNieve, H. L., III, M.S. thesis, Univ. of Tennessee, Knoxville (1963).
11. LaNieve, H. L., III, and D. C. Bogue, *J. Appl. Polym. Sci.*, **12**, 353 (1968).
12. Ramamurthy, A. V., Ph.D. thesis, Monash Univ., Melbourne, Australia (1970); also paper with D. V. Boger, *Trans. Soc. Rheol.*, **15**, 709 (1971).
13. Sylvester, N. D., and S. L. Rosen, *AIChE J.*, **16**, 964 (1970).
14. Uebler, E. A., Ph.D. thesis, Univ. of Delaware, Newark (1966).
15. Vrentas, J. S., J. L. Duda, and K. G. Bargeron, *AIChE J.*, **12**, 837 (1966).
16. Wang, Y. L., and P. A. Longwell, *AIChE J.*, **10**, 323 (1964).
17. Weissenberg, H. L., *J. Phys. of Fluids*, **5**, 1033 (1962).

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Stability of Time-Delay Systems

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A direct method is presented for determining both local and regional stability of systems described by nonlinear differential-difference equations. Prediction of stability is with respect to a general class of initial curves. The practical as well as the conservative nature of the procedure is demonstrated by a numerical example.

The dynamics of many physical systems may be represented by differential-difference equations of the form

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{x}(t-\boldsymbol{\theta})] \qquad (1)$$

$$\mathbf{f}[\mathbf{0}, \mathbf{0}] = \mathbf{0}$$

where $\mathbf{x}$, $\mathbf{x}(t-\boldsymbol{\theta})$ and $\mathbf{f}$ are vectors of dimension n and the $\boldsymbol{\theta}$ are the values of m time delays. In chemical engineering systems, this form can arise, for example, when the dynamics of the units of a process are represented by differential equations, but the total process includes transport lags between the units. It occurs also when a unit is being controlled via a time-delayed feedback. Such systems can have multiple time delays, nonlinearities, and more than one equilibrium state.

The state of a system described by Equation (1) depends on the value of $\mathbf{x}(t-\boldsymbol{\theta})$, the state variables at some previous times; that is to say, the state will have a continuous dependence on the past history of the system. The solution of Equation (1) is uniquely defined only when initial (past history) curves are specified. In this system initial curves for Equation (1) are analogous to initial conditions in the study of differential equations. Such curves may be defined by

$$\mathbf{x}(t) = \mathbf{x}'(t), \ \ (t_0 - \boldsymbol{\theta}) \leq t \leq t_0 \qquad (2)$$

or equivalently

$$\mathbf{x}(t-\boldsymbol{\theta}) = \mathbf{x}'(t), \ t_0 \leq t \leq (t_0 + \boldsymbol{\theta}) \qquad (3)$$

where t_0 represents an initial time, from which equation (1) is to represent the dynamics of the system.

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Except for some very special cases, analytic solutions of Equation (1) are not available, a circumstance that makes any direct assessment of stability an especially attractive alternative. To this end Seborg and Johnson (1) have studied the stability of nonlinear systems containing time delays by making use of Liapunov functions. After expressing the time derivative of the Liapunov function as explicitly dependent on the $\mathbf{x}(t)$ and $(d\mathbf{x}/dt)$, these authors established sufficient sign-definite conditions in terms of the maximum values of the $(d\mathbf{x}/dt)$. The stability results to be presented in this paper were also obtained via a Liapunov-function argument; the search for maximum values of the $(d\mathbf{x}/dt)$ is however avoided by making use of the theorems developed by Razumikhin (*2* to *4*). By coupling this method with techniques for determining an allowable set of initial curves, widely applicable algorithms are developed for examining both local and regional stability.

STABILITY CRITERIA

Razumikhin postulated a set of stability conditions which rely on bounding the magnitude of $\mathbf{x}(t - \boldsymbol{\theta})$ by a function composed only of the components of $\mathbf{x}(t)$. This allows the stability analysis of the system to be performed, even though the exact relationship between $\mathbf{x}(t)$ and $\mathbf{x}(t - \boldsymbol{\theta})$ is not determined.

Consider a system obeying the dynamics of Equation (1), and an associated Liapunov function $v[\mathbf{x}(t)]$, having as time derivative the functional $dv/dt = \dot{v} = W[\mathbf{x}(t), \mathbf{x}(t - \boldsymbol{\theta})]$. Let $\mathbf{y}(t)$ be the set of continuous curves over which $W[x(t), x(t - \boldsymbol{\theta})]$ is defined. The asymptotic stability of such a system can be assured if all the curves contained within the $[\mathbf{y}(t)]$ are bounded and approach the equilibrium point as $t \to \infty$. Razumikhin's theorems provide the criteria for this restriction by exploiting the fact that all possible system behavior must fall into one of three categories. Either

$$v[\mathbf{y}(\sigma)] > v[\mathbf{y}(t)]\ ,\ \sigma < t \qquad (4)$$

for all time and all $\mathbf{y}(t)$; or

$$v[\mathbf{y}(\sigma)] \leqq v[\mathbf{y}(t)]\ ,\ \sigma < t \qquad (5)$$

for all time; or lastly, the functions $[\mathbf{y}(t)]$ could at different times satisfy either inequality (4) or (5). Formal proof and details may be found in (*1, 2*). In essence, two approaches are suggested as means for determining stability.

To determine stability via the first approach, prediction of stability is with respect to the general function $[\mathbf{y}(\sigma)]$ and is independent of the magnitude of the delay. Less stringent stability conditions are obtained by the second method in which the set of curves $[\mathbf{y}(t)]$ is restricted to only solutions of Equation (1). For such a purpose, let S_{j1} and S_{j2} be bounds on the magnitude of the derivatives of Equation (1), that is,

$$S_{j1}(c) = \max_{||\mathbf{y}|| < c} \{f_j[\mathbf{y}(t),\ \mathbf{y}(t - \boldsymbol{\theta})]\} \qquad j = 1, 2, \ldots, n \qquad (6)$$

$$S_{j2}(c) = \min_{||\mathbf{y}|| < c} \{f_j[\mathbf{y}(t),\ \mathbf{y}(t - \boldsymbol{\theta})]\} \qquad (7)$$

on the set of continuous curves satisfying the conditions

$$v[\mathbf{y}(\sigma)] \leqq v[\mathbf{y}(t)] = c,\ (t - \boldsymbol{\theta}_m) \leqq \sigma \leqq t \qquad (8)$$

It then follows from Equation (1) that

$$|t_1 - t_2|S_{j2} \leqq |x_j(t_1) - x_j(t_2)| \leqq S_{j1}|t_1 - t_2|;$$
$$j = 1, \ldots, n;\ (t - \theta_m) \leqq t_1 \leqq t;\ \ (t - \theta_m) \leqq t_2 \leqq t. \quad (9)$$

The Razumikhin second method involves delays explicitly and offers the possibility of estimating their influence on stability by means of the theorem:

If, for the set of Equation (1), there exists a finite positive-definite function $v[\mathbf{x}(t)]$ and a positive number c such that the derivative of the function v determines the functional $\dot{v} = W[\mathbf{x}(t),\ \mathbf{x}(t - \boldsymbol{\theta})]$ through Equation (1) where this functional is negative-definite along every curve satisfying Equation (8) for $t \geqq t_0$ and Equation (9) for $(t - \theta_m) \leqq t_1 \leqq t \geqq t_0$, $(t - \theta_m) \leqq t_2 \leqq t \geqq t_0$, then the steady state is asymptotically stable.

INITIAL CURVES

The solution of Equation (1) is continuously dependent upon the initial curves, a property reflected in the stability criteria as presented in the theorem. As a consequence, any result will be with respect to a set of initial curves. It is desirable not to place any restrictions on the initial curves other than that of being contained within a given region, but some means will be needed for determining the constraint region:

$$||x(t_0 - \theta_m)|| \leqq \beta(c, T)\ ;\ T > (t_0 + \theta_m) \qquad (10)$$

such that the validity of $v[x(t)] \leqq c$ in the same interval $(t_0,\ T)$ insures asymptotic stability by the conditions of the theorem.

In their studies of the stability of differential equations Weiss and Infante (5) discuss methods of relating positive-definite functions over finite times. Following their approach let v_0 be a positive number such that

$$v[\mathbf{x}(t_0)] \leqq v_0 = \beta(c, T) \qquad (11)$$

and let λ be defined as

$$\lambda(t) = \frac{v[\mathbf{x}(t), \mathbf{x}(t - \boldsymbol{\theta})]}{v[\mathbf{x}(t)]} \qquad (12)$$

It has been shown (*7*) that v_0 can be an upper bound for a positive definite function

$$v[\mathbf{x}(t_0 + \theta_m)] \exp(-\lambda_m \theta_m) \leqq v_0 \qquad (13)$$

where λ_m is defined as the maximum value of $\lambda(t)$ for all $\mathbf{x}(t)$ contained in the hyperspace between the contours $v = v_0$ and $v = c$ and for all $\mathbf{x}(t - \theta)$ contained within the union or sum of the regions enclosed by v_0 and c:

$$\lambda_m = \mathrm{Max}\left\{ \frac{v[\mathbf{x}(t), \mathbf{x}(t - \boldsymbol{\theta})]}{v[\mathbf{x}(t)]} \right\} \qquad (14)$$

If in the stability analysis a region is determined such that

$$v[\mathbf{x}(t_0 + \boldsymbol{\theta}_m)] \leqq c$$

then equation (13) can be used to determine the positive number v_0 and hence the contour boundary $v[\mathbf{x}(t_0)] \leqq v_0$ for the initial curves.

EXAMPLE

Luyben (*6*) developed a mathematical model for the autorefrigerated reactor in which heat generated by an exothermic reaction is removed by an endothermic change in the state of the solvent. The reactor system consists of a continuous-flow stirred-tank reactor (CSTR) in which an irreversible first-order reaction is occurring. A solvent is cycled through a refrigeration system where the latent

The term *undisturbed motion* is common in the Russian literature. For the purposes of this paper, it is essentially the same as *steady state*, though it can have broader connotation in other problems.

heat of vaporization is removed. It is assumed that the liquid which is returned is not subcooled. The mass and energy balances for the reactor are

$$V \frac{dC}{dt} = q(C_0 - C) - VR \quad (15)$$

$$VC_p \frac{dT}{dt} = qC_p(T_0 - T) + \Delta HVR - \Delta H_v q_c \quad (16)$$

where

$$R(C, T) = k_0 C \exp(-Q/T) \quad (17)$$

The latent heat of vaporization is assumed to be a parabolic function of temperature

$$\Delta H_v = \begin{cases} \Delta H_v^0 [1 - (T/T_c)]^{1/2}, & T < T_c \\ 0 & , T \geq T_c \end{cases} \quad (18)$$

The reactor is to be controlled by measuring the reactor temperature and regulating the vapor stream flow rate by means of simple proportional control.

$$q_c - q_R = \hat{q} = K_p (T - T_R) = K_p \hat{T} \quad (19)$$

Equation (19) implies that a temperature disturbance in the reactor can be instantaneously transmitted to a change in flow rate. In a real system, a finite time is required to transfer such information. If the temperature perturbation varies slowly in comparison to the response of the controller, the dynamics of the control loop can be approximated by pure delay (7) and

$$\hat{q}(s) = K_p \exp(-s\theta)\, \hat{T}(s) \quad (20)$$

In the time domain, Equation (20) is expressed as

$$q_c - q_R = K_p[T(t - \theta) - T_R] \quad (21)$$

By replacing Equation (19) with Equation (21) in the representation of the autorefrigerated reactor, the dynamics for the controller have been included in the form of a single parameter θ.

The three steady states for the system are given in (6) for the values of the parameters chosen by Luyben. The values of the first and third steady states are independent of controller gain; however, the stability of the first is dependent upon the magnitude of the controller gain, as well as on θ, the time delay associated with the controller dynamics.

LOCAL STABILITY

The equations for the reactor system can be written in dimensionless form by defining reduced deviation variables in the usual way (7) to obtain

$$\frac{dx_1}{dt} = \frac{-q\, x_1}{V} - \frac{r}{C_0} \quad (22)$$

$$\frac{dx_2}{dt} = \frac{-q\, x_2}{V} + \frac{r}{C_0} - d_1 \left\{ [q_R + d_2 x_2(t - \theta)] \left[\sqrt{1 - \frac{x_2 + \eta_s}{\eta_c}} - q_R \sqrt{1 - \frac{\eta_s}{\eta_c}} \right] \right\} \quad (23)$$

By linearization about the steady state

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \quad (24)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + a_{23}x_2(t - \theta) \quad (25)$$

The symmetric quadratic Liapunov function

$$v = \mathbf{x}^T\mathbf{P}\mathbf{x} \quad (26)$$

has the derivative

$$\dot{v} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = 2\dot{\mathbf{x}}^T\mathbf{P}\mathbf{x}$$

The latter becomes by substitution of Equations (24) and (25)

$$\dot{v} = \mathbf{x}^T(\mathbf{PA} + \mathbf{A}^T\mathbf{P})\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{A}_1\mathbf{x}(t - \theta) + \mathbf{x}^T(t - \theta)\mathbf{A}_1^T\mathbf{P}\mathbf{x} \quad (27)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad ; \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & a_{23} \end{bmatrix}$$

It is convenient to replace the x variables in Equation (27) according to the linear transformation:

$$\mathbf{x} = \mathbf{C}\mathbf{u} \quad (28)$$

where $\mathbf{C}$ is such that the v function is a circle in the space of $\mathbf{u}$. Razumikhin has demonstrated how his theorem can be used to obtain an upper bound on v as an explicit function of θ and y. Applying the procedure and choosing the elements $\mathbf{P}$ and $\mathbf{C}$ to be:

$$\mathbf{P} = \begin{bmatrix} -a_{21}/a_{12} & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{C} = \begin{bmatrix} (-a_{12}/a_{21})^{1/2} & 0 \\ 0 & 1 \end{bmatrix}$$

results in the following inequality

$$\dot{v} \leq \dot{v}_1 = \mathbf{u}^T\mathbf{B}\mathbf{u} \quad (29)$$

where

$$b_{11} = 2a_{11} + 2\theta|a_{23}|[(-a_{21}a_{12})^{1/2} + |a_{22}| + |a_{23}|]$$

$$b_{12} = b_{21} = 0$$

$$b_{22} = 2(a_{22} + a_{23}) + 2\theta|a_{23}|[(-a_{21}a_{12})^{1/2} + |a_{22}| + |a_{23}|]$$

Since $\mathbf{B}$ is diagonal, asymptotic stability is guaranteed if

$$b_{11} < 0, \quad b_{22} < 0 \quad (30)$$

The curve labeled sepatrix for Liapunov analysis in Figure 1 corresponds to a locus of (K_p, θ) values which satisfy

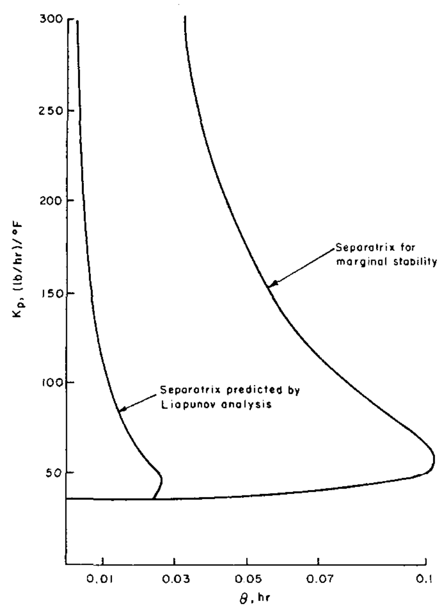

Fig. 1. Parametric study of the effect of gain and controller dynamics on the stability of a controlled autorefrigerated reactor.

$b_{11} = b_{22} = 0$. Combinations of (K_p, θ) to the left of this curve therefore satisfy the inequalities in Equation (39) and correspond to operating conditions for which the system is asymptotically stable.

To assess the conservatism of the Liapunov approach to this problem, a second independent stability analysis was performed by using the classical Nyquist criterion. The results of the analysis are also illustrated in Figure 1. Since the Nyquist analysis is based on both sufficient and necessary conditions for stability, the resulting sepatrix corresponds to a locus of marginally stable systems. As the size of θ increases, the choice of K_p is restricted until for $\theta \geq 0.1$ hr. the system can no longer be operated at this equilibrium point. Evidently the controller dynamics can have considerable influence on the controller gain, even when the controller dynamics are fast in comparison with the reactor time constants. Since the Liapunov method is based on sufficient but not necessary conditions for stability, the distance between the loci is to be expected. Note, however, that over the range of typical well-designed control, (θ from say 0.003 to 0.008 hr.) the Liapunov method correctly and precisely predicts a wide range of controller gain which would result in a stable operation.

When evaluating two methods for predicting stability, one point for comparison is the relative effort necessary to apply the methods. In this case, the Liapunov method is easier to apply because the calculation of b_{11} and b_{22} can be accomplished with the aid of a desk calculator while the Nyquist analysis required more extensive computer calculations.

REGIONS OF STABILITY

In order to study the stability of a system when it is subject to finite disturbances, the original nonlinear equations must be analyzed. The formulation of the problem is similar to the Liapunov method developed for the linear analysis except that the functional v will no longer be a simple quadratic form.

For the sake of clarity, let the right-hand sides of Equations (25) and (26) be expressed as

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}, \mathbf{x}(t-\theta)]$$

The same Liapunov function as in Equation (26) will here produce the time derivative

$$\dot{v} = \mathbf{x}^T\mathbf{P}\mathbf{f}[\mathbf{x}, \mathbf{x}(t-\theta)] + \mathbf{f}^T\mathbf{P}\mathbf{x} \tag{31}$$

The coefficient matrix $\mathbf{P}$ and the transformation matrix $\mathbf{C}$ will be the same as was used in the linear analysis. By again substituting for $\mathbf{x}$ Equation (31) becomes

$$\dot{v} = \mathbf{u}^T\mathbf{C}^T\mathbf{P}\mathbf{f}[\mathbf{u}, \mathbf{u}(t-\theta)] + \mathbf{f}^T\mathbf{P}\mathbf{C}\mathbf{u} \tag{32}$$

There exists one delay variable $x_2(t-\theta)$ in Equations (22) and (23). For the $\mathbf{C}$ chosen

$$u_2(t-\theta) = x_2(t-\theta)$$

and the mean value theorem for $u_2(t-\theta)$ gives

$$u_2(t-\theta) = u_2(t) - \theta\dot{u}_2(\sigma), \quad (t-\theta) \leq \sigma \leq t \tag{33}$$

The mean value theorem can be used to eliminate the delay variables from Equation (32)

$$\dot{v} = \mathbf{u}^T\mathbf{C}^T\mathbf{P}\mathbf{f}[\mathbf{u}, \theta, \dot{u}_2(\sigma)] + \mathbf{f}^T\mathbf{P}\mathbf{C}\mathbf{u} \tag{34}$$

To obtain the upper and lower bounds

$$S_L \leq \dot{u}_2(\sigma) \leq S_u \tag{35}$$

Equations (6) and (7) are used with appropriate sign choices to assure that

$$S_{j1} \leq S_u \tag{36}$$

$$S_{j2} \geq S_L \tag{37}$$

where S_{j1} and S_{j2} are explicit functions of the $\mathbf{u}(t)$. Substitution in Equation (34) then yields

$$\dot{v} \leq \dot{v}_1 = \mathbf{u}^T\mathbf{C}^T\mathbf{P}\mathbf{f}[\mathbf{u}(t), \theta, S_L, S_u] + \mathbf{f}[\mathbf{u}(t), \theta, S_u, S_L]^T\mathbf{P}\mathbf{C}\mathbf{u} \tag{38}$$

Equation (38) has $\dot{v}_1$ as a function (not a functional) and may be studied by the established techniques for determining a region in which $v_1 < 0$, however a region containing permissible initial curves must also be obtained. An upper bound on λ_m is determined by replacing $\dot{v}$ by $\dot{v}_1$ in Equation (14):

$$\lambda_m \leq \underset{||\mathbf{u}|| \leq c}{\text{Max}} \left\{ \frac{\dot{v}_1[\mathbf{u}(t)]}{v[\mathbf{u}(t)]} \right\} \tag{39}$$

From Equation (13)

$$v_0 \leq v[\mathbf{u}(t)] \exp(-\theta\lambda_m) \tag{40}$$

In the event that it is not possible to find a region $v \leq c$ everywhere within which $\dot{v}_1$ in Equation (38) is negative, the problem can be circumvented by determining contractive stability instead of asymptotic stability. Whereas asymptotic stability requires that the solution trajectories are bounded and approach the equilibrium state, contractive stability requires only that the trajectories are eventually confined to a region $||\mathbf{x}|| < \beta_1$ containing the equilibruim state. By choosing this region to contain all points in which $\dot{v}_1$ is not negative, contractive stability is established for the system. Because no information about the dynamics within β_1 is obtained in determining contractive stability, the specifications of the size of β_1 is an important parameter which must be chosen so that the results of the analysis have practical engineering significance.

To obtain a region of contractive stability for the reactor system, search procedures were used to determine λ_m and $\dot{v}_1$. The maximum value of $\dot{v}_1$ is

$$\dot{v}_m = \underset{\beta_1 \leq ||\mathbf{u}|| \leq c}{\text{Max}} [\dot{v}_1] \tag{41}$$

for chosen values of c and β_1. For this example β_1 was chosen so that the maximum temperature and concentration deviation from the equilibrium state contained within β_1 would be 2°F. and 0.02 lb./lb. A grid search was used to determine the sign of $\dot{v}_m$. Once a region of contractive stability was obtained, the search for λ_m was initiated.

Results of a regional stability analysis are shown in Figures 2 and 3. In Figure 2 the region enclosed by the contour $v = 1.226 \times 10^{-2}$ is a region of contractive stability when the initial curves are contained within the contour $v_0 = 0.078 \times 10^{-2}$; that is, solutions of Equations (22) and (23), following any initial disturbances of concentration and temperature within the contour v_0, are bounded by the contour v and will degenerate to the region contained within the contour β_1. The remaining curve on this figure is the marginal stability sepatrix obtained by Luyben for the reactor system without the controller time lag. A comparison of these results must recognize that the Luyben stability analysis without time delay was based on sufficient and necessary conditions. The reduction of size of the region of stability could therefore be the effect of either time delay or the stability method or both.

A more realistic analysis of the effect of θ on the size of the region of stability can be obtained by comparing regions determined by applying the same stability criteria for different values of θ, for a specified region of contractive stability. Figure 3 shows that stability sepatrices can be found in the (K_p, θ) plane corresponding to particular regions of contractive stability. The region in the plane above and to the left of these sepatrices represents combinations of (K_p, θ) for which stability is insured. As the size of the region of stability in state space is increased, the area in parameter space enclosed by the corresponding sepatrix is decreased. For a fixed controller tuning K_p, the size of the region of contractive stability is decreased as the time lag θ is increased. Comparison of Figures 1 and 3 shows that the effect of the (K_p, θ) parameters on regional stability is similar to that obtained in the local stability analysis. As expected, the sepatrices of the regional stability analysis are to the left of both local stability sepatrices, since the regional stability predictions are more conservative.

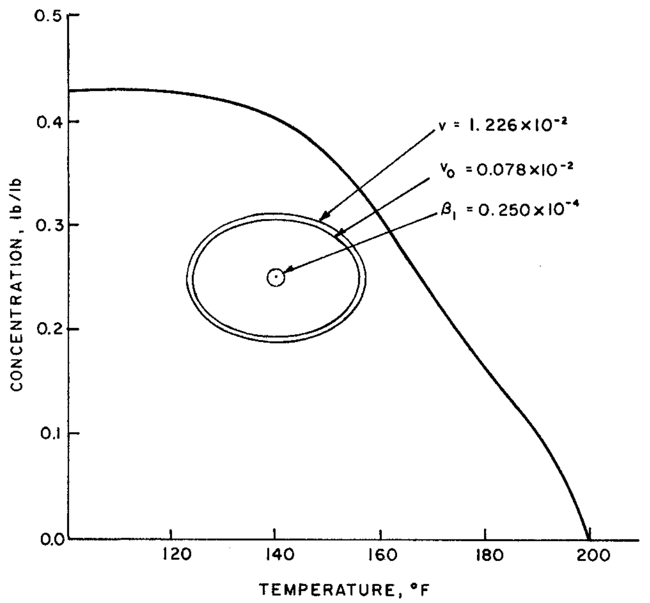

Fig. 2. Regions of stability for autorefrigerated reactor, $K_p = 100$ lb./hr. °F., $\theta = 0.004$ hr.

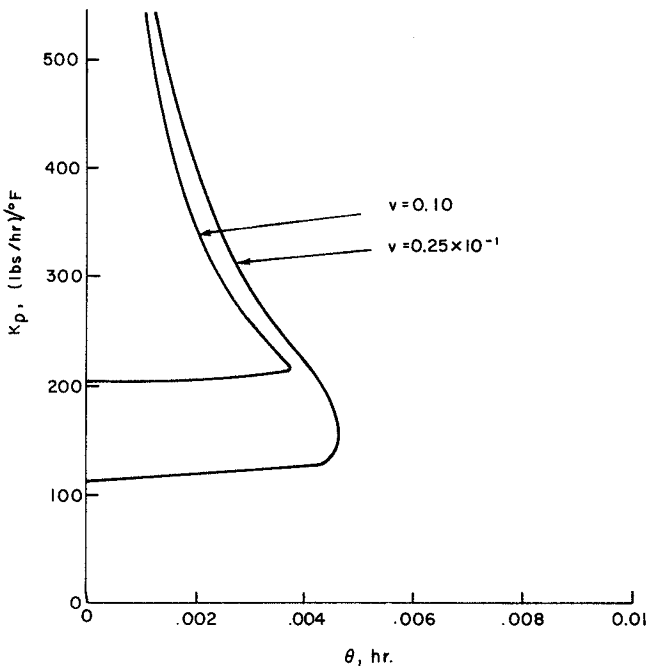

Fig. 3. Parametric study of the effect of gain and controller dynamics on the regional stability of an autorefrigerated reactor.

ACKNOWLEDGMENT

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NOTATION

- $\mathbf{C}$ = a transform matrix for Liapunov function
- C = reactant concentration
- C_p = specific heat
- d_1 = $\Delta H_v^0 / V \Delta H C_0$
- d_2 = $K_p \Delta H C_0 / C_p$
- ΔH = heat of reaction
- ΔH_v = latent heat of vaporization
- k_0 = frequency factor
- K_p = proportional gain of the controller
- $\mathbf{P}$ = quadratic coefficient matrix for Liapunov function
- Q = activation energy/gas constant
- q = mass flow rate
- R = reaction rate
- r = deviation of reaction rate from the steady state value, $(R - R_s)$
- S_{j1}, S_u = upper bounds
- S_{j2}, S_L = lower bounds
- T = temperature, or specified time
- t = time
- t_0 = initial time from which Equation (1) applies
- $\mathbf{u}$ = transform variable
- V = reactor volume
- v = Liapunov function
- $\dot{v}_1$ = upper bound on time derivative of v
- $\mathbf{x}$ = general state vector of dimension n
- $\mathbf{x}'$ = value of state vector in time interval $(t_0 - \theta) \leq t \leq t_0$
- x_1 = deviation of dimensionless concentration, $(y - y_s)$
- x_2 = deviation of dimensionless temperature, $(\eta - \eta_s)$
- y = curves in x state vector space, or C/C_0
- η = $(C_p/\Delta H C_0) T$
- λ = ratio of the Liapunov function to its derivative
- λ_{max} = maximum value of λ in some space of $\mathbf{x}$
- $\boldsymbol{\theta}$ = time delay vector
- θ_m = maximum value of time delays
- a_{11} = $-q/V - k_0 \exp(-Q/(T_s + 460))$
- a_{12} = $-k_0 y_s \Delta H C_0 Q \exp(-Q/(T_s + 460))/C_p$
- a_{21} = $k_0 \exp(-Q/(T_s + 460))$
- a_{22} = $-q/V - a_{12} + d_1 C_p q_R (1 - \eta_s/\eta_c)^{1/2}/2\eta_c$
- a_{23} = $-\Delta H_v^0 K_p (1 - \eta_s/\eta_c)^{1/2}/V C_p$

Subscripts-Superscripts

- 0 = reactor inlet conditions
- R = control or reference setting
- s = equilibrium state
- ϵ = contained within
- $\cdot$ = total time derivative
- $\wedge$ = deviation from reference point
- c = critical temperature

LITERATURE CITED

1. Seborg, D. E., and E. F. Johnson, *AIChE J.*, **17**, 872 (1971).
2. Razumikhin, B. S., Ph.D. thesis, Inst. Medhaniki, Akad. Nauk. USSR (1958).
3. ———, *Applied Math. Mech.*, **22**, 215 (1959).
4. ———, *Auto. Remote Control*, **21**, 515 (1960).
5. Weiss, L., and E. F. Infante, *Proc. Natl. Acad. Sci.*, **54**, 44 (1965).
6. Luyben, W. L., *AIChE J.*, **12**, 662 (1966).
7. Landis, J. G., Ph.D. thesis, Univ. Pennsylvania, Philadelphia (1970).

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